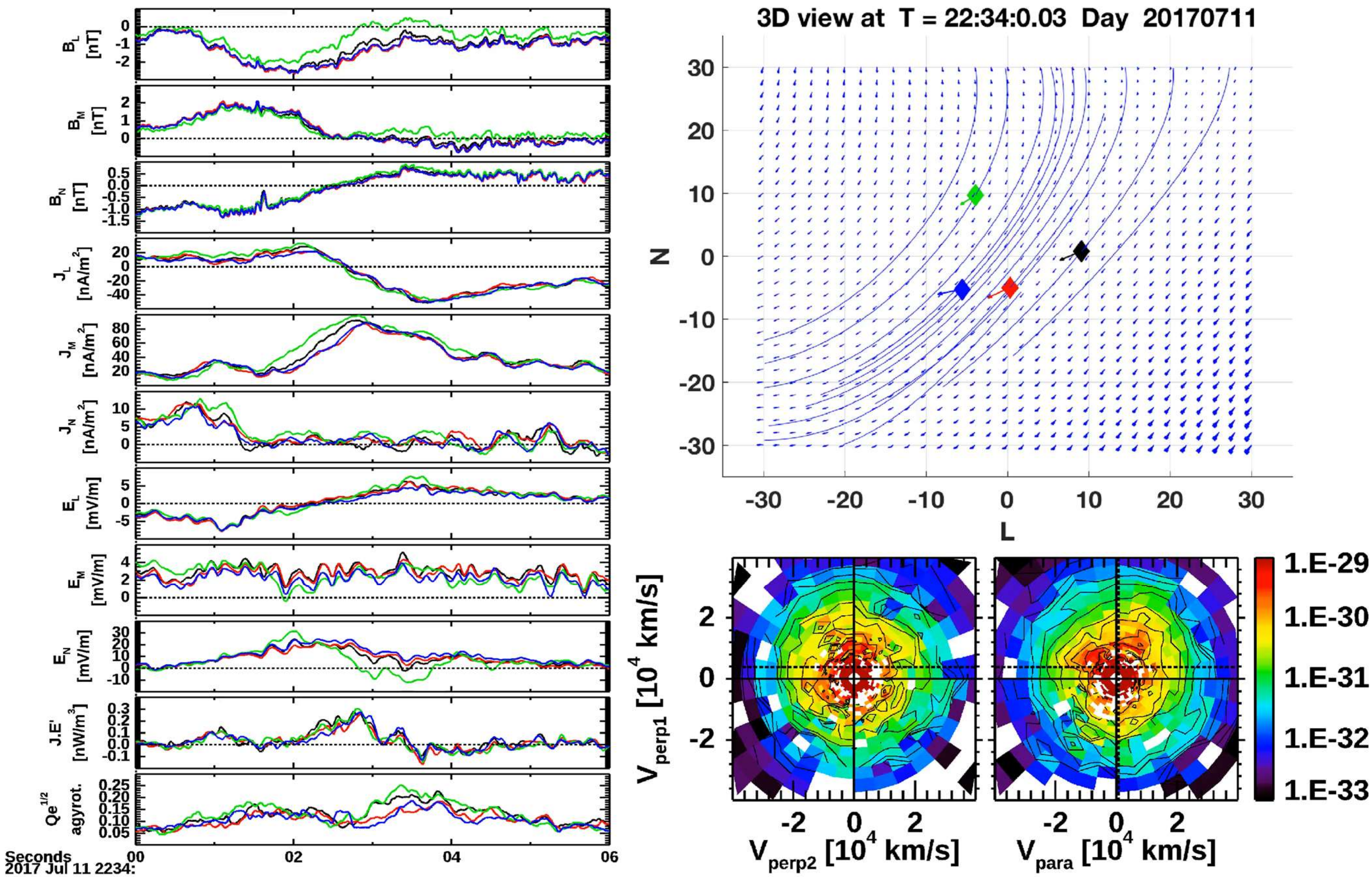
2nd Order Field Reconstruction: Method and Applications

Ivan Dors, R. B. Torbert, M. R. Argall,
K. J. Genestreti, R. E. Denton,
D. Payne, R. Strangeway, R. E. Ergun,
B. L. Giles, J. L. Burch



3D view at T = 22:34:0.03 Day 20170711
N
L
B_L [nT]
B_M [nT]
B_N [nT]
J_L [nA/m^2]
J_M [nA/m^2]
J_N [nA/m^2]
E_L [mV/m]
E_M [mV/m]
E_N [mV/m]
J.E' [nW/m^3]
$Qe^{1/2}$ agyrot.
Seconds
2017 Jul 11 2234:
V_{perp1} [10^4 km/s]
V_{perp2} [10^4 km/s]
V_{para} [10^4 km/s]
f (s^3/cm^6)
1.E-29
1.E-30
1.E-31
1.E-32
1.E-33

Goal

- Reconstruct the 3D $\vec{B}$ field to 2nd-order polynomial using the $\vec{B}$ and $\vec{J}$ at 4 s/c locations.

1st Order

$$B_i{}'(\vec{x}) \cong \underbrace{\overbrace{B_i\Big|_0 + \sum_k \partial B_{ki}\Big|_0 x_k}^{\text{1st Order}} + \frac{1}{2}\sum_{kl} \partial\partial B_{kli}\Big|_0 x_k x_l}_{\text{2nd Order}}$$

2nd Order

1st-Order Parameter Overview

- Knowns: **12**

  3 $\vec{B}$ components x 4 s/c → 12

- Unknowns: **12**

  B_i → 3 Elements of $\vec{B}$ at origin

  ∂B_{ij} → 3x3 = 9 Elements of the $\nabla\vec{B}$ tensor about origin

- Constraints: **0**

1B1		1	1X	1Y	1Z										B1
2B1		1	2X	2Y	2Z										dB11
3B1		1	3X	3Y	3Z										dB21
4B1		1	4X	4Y	4Z										dB31
1B2						1	1X	1Y	1Z						B2
2B2	=					1	2X	2Y	2Z					x	dB12
3B2						1	3X	3Y	3Z						dB22
4B2						1	4X	4Y	4Z						dB32
1B3										1	1X	1Y	1Z		B3
2B3										1	2X	2Y	2Z		dB13
3B3										1	3X	3Y	3Z		dB23
4B3										1	4X	4Y	4Z		dB33

2^{nd}-Order Parameter Overview

- Knowns: **24**
  3 $\vec{B}$ components x 4 s/c→12
  3 $\vec{J}$ components x 4 s/c→12

- Unknowns: **27**

B_i→ 3	Elements of $\vec{B}$ at origin
∂B_{ij}→ 3x3 = 9	Elements of the $\nabla\vec{B}$ tensor about origin
$\partial\partial B_{ijk}$→ 3x3x3 = 27	Elements of the $\nabla\nabla\vec{B}$ tensor about origin
$\partial\partial B_{ijk} = \partial\partial B_{jik}$→ -9	Clairaut's theorem for continuous functions
$\partial\partial B_{22k} = 0$ → -3	Assume linear behavior in M-direction (NML system)

- Constraints: **4**

$\nabla \cdot \vec{B} = 0$ → 4	1 for traceless $\nabla\vec{B}$ tensor; 3 to be true in all space: $\nabla(\nabla \cdot \vec{B})$ = 0

24 + 4 - 27 = 1 more unknown (degree of freedom) is allowed.
A cubic term is incorporated to make the system solvable under all conditions.
8 unique cubic terms are used without affecting $\nabla \cdot \vec{B}$
Each produces a similar result.

Reconstruction Method

(Knowns) = (Taylor approximation with Constraints) (Unknowns)

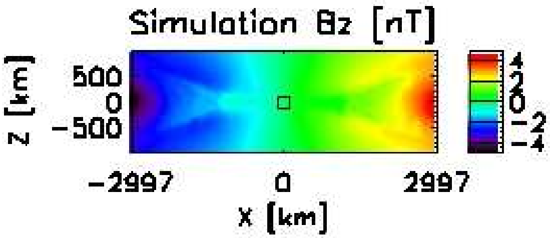
- The linking matrix is between 24x24 to 31x31 depending on how the constraints are applied.
- The matrix is inverted to solve for the Taylor expansion coefficients.
- This is repeated for each of the 8 unique cubic terms.
- The 8 results are combined into one set of coefficients weighted inversely by the magnitude of the cubic parameter.
- The resulting coefficients allow for the approximation of $\vec{B}(\vec{x})$ & $\vec{J}(\vec{x})$

$$B_i'(\vec{x}) \cong B_i\Big|_0 + \sum_k \partial B_{ki}\Big|_0 x_k + \frac{1}{2}\sum_{kl} \partial\partial B_{kl}\Big|_0 x_k x_l$$

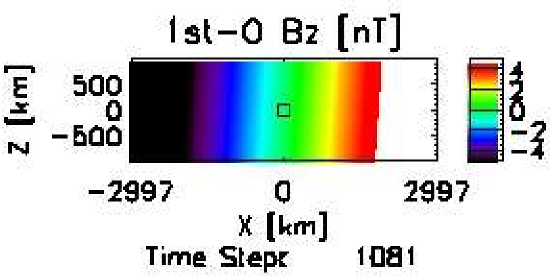
Limitations

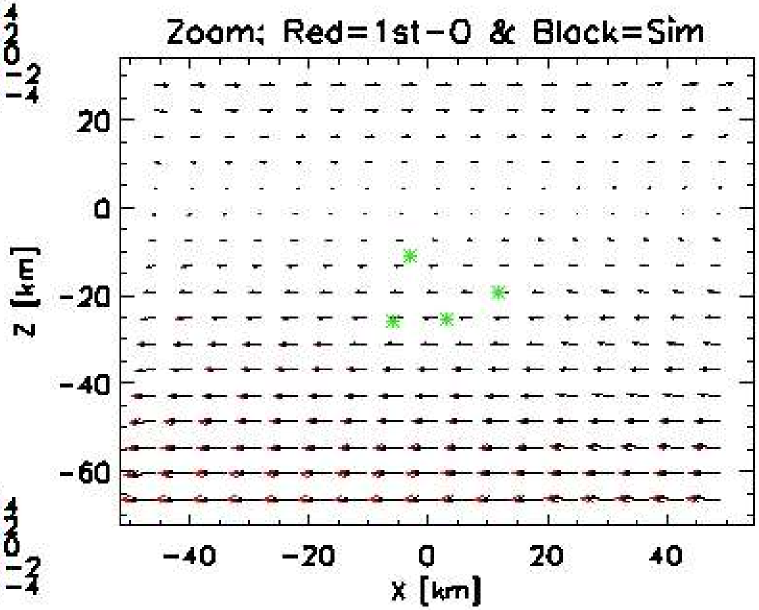
- Results degrade if the basis functions do not describe the fields.
  - Small structures unresolved by the s/c greatly affect the results
  - This is partially addressed by temporal filtering to focus on the spatial scale of interest: $\tau \sim l_{sc} / v_{structure}$
- Results degrade as approximations are made further from the s/c.
  - Extrapolation errors are expected and are unavoidable
  - Reconstruction of simulation data helps quantify this limitation
- These limitations are not unique to this method.

1st Order B Result Using Simulation



↕ **Good global fit**





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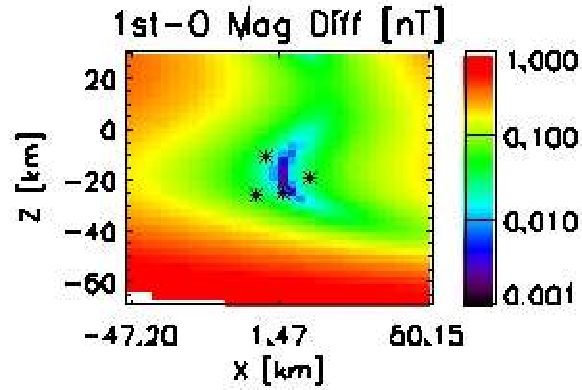
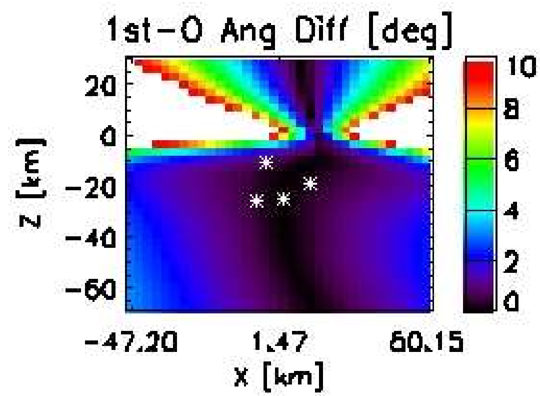


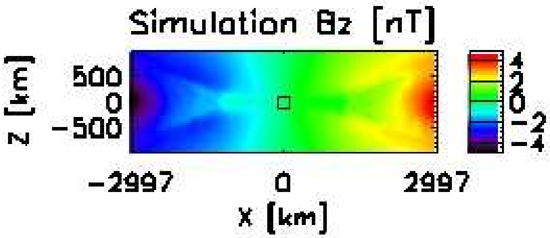


<u>Fitting to B only</u>

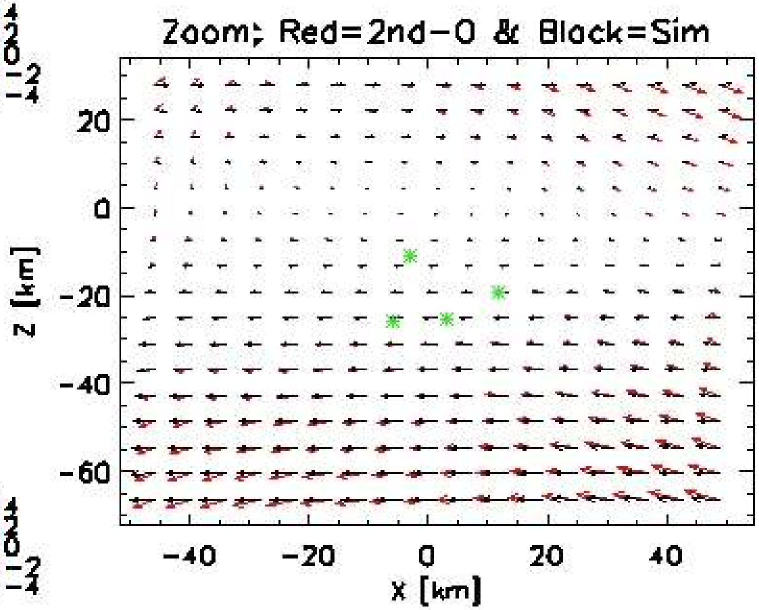
$$\nabla \cdot \vec{B} \neq 0$$

$$\nabla \times \vec{B} \neq \mu_0 \vec{J}$$

2^{nd} Order B Result Using Simulation



↕ **Strongly focused**



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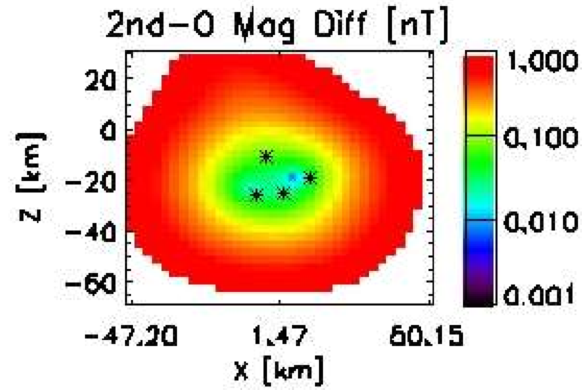
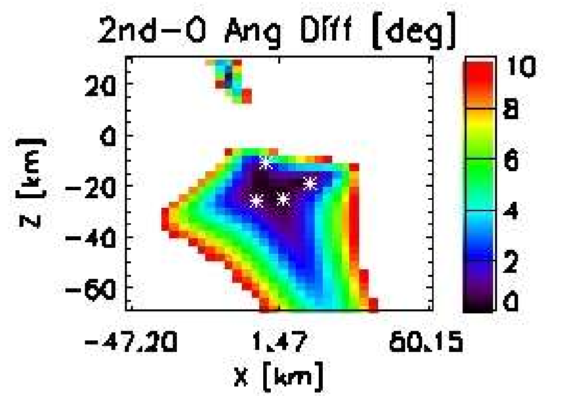


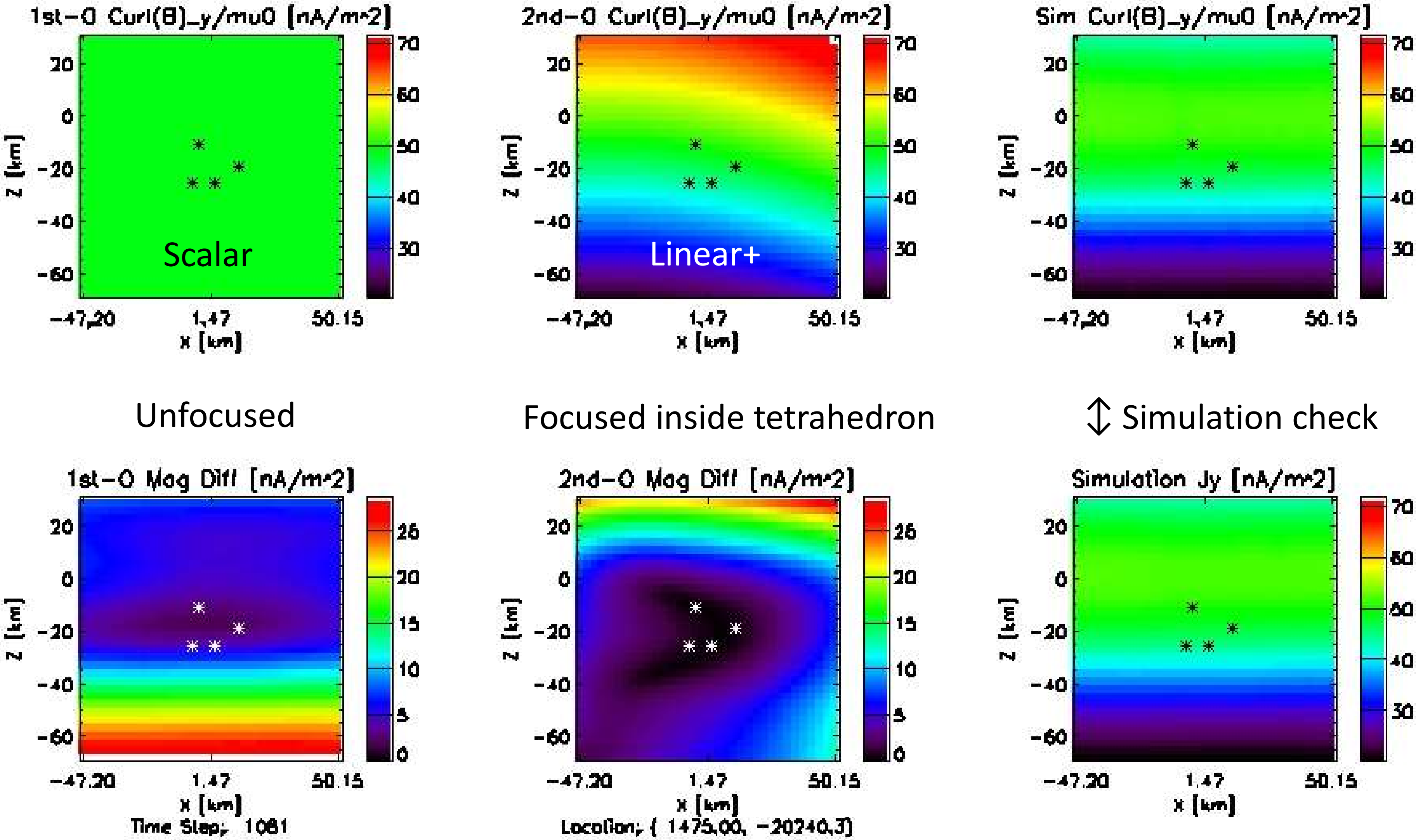


<u>Fitting to B & J</u>

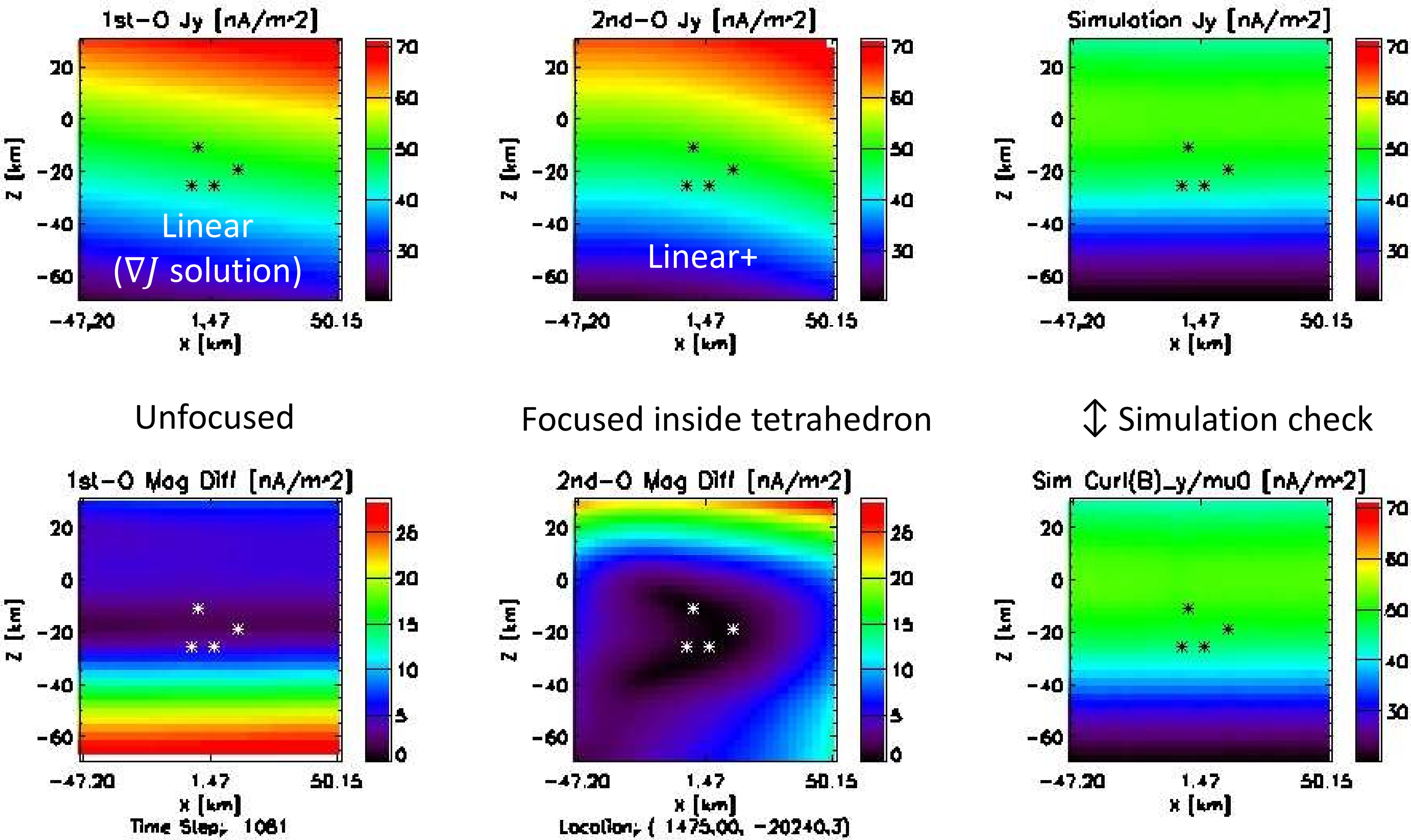
$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

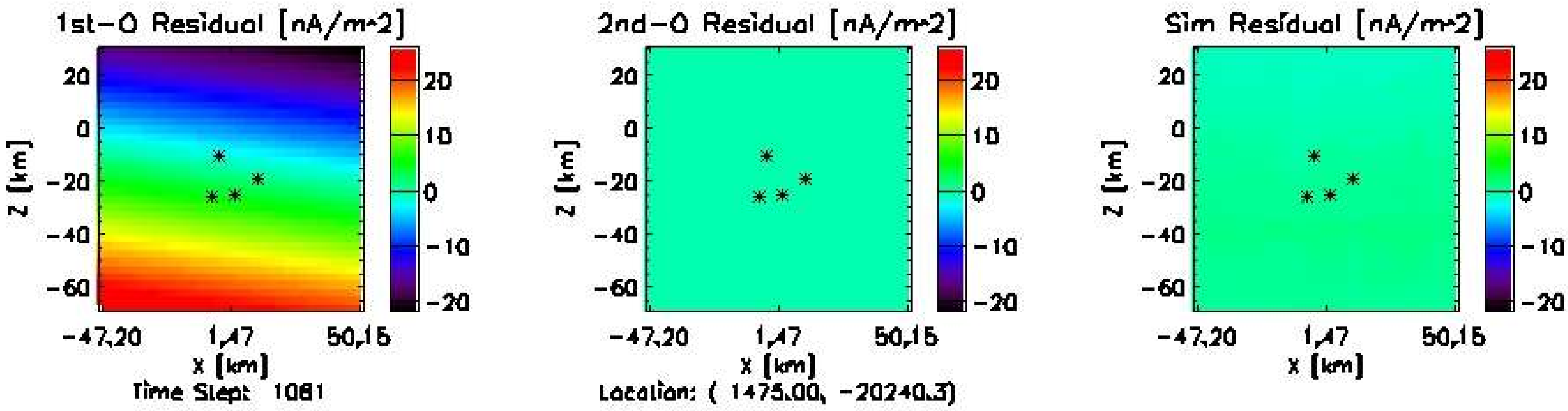
Curl(B)/μ_o Comparison



J Reconstruction Comparison



$\left(\nabla \times \vec{B}/\mu_o - \vec{J}\right)$ Comparison



1st order B & J results violate Ampere's law, even at the barycenter.
Simulation and 2nd order reconstruction obey Ampere's law.

$\vec{E}$ Field Reconstruction

4 $\vec{B}$
4 $\vec{J}$ ⇨

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\nabla \cdot \vec{B} = 0$$

⇨ $\left[\vec{B}\right]_q$ $\left[\vec{J}\right]_l$

Knowns | Link Matrix | Unknowns

4 $\vec{E}$
4 $\dot{\vec{B}}$ ⇨

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

⇨ $\left[\vec{E}\right]_q$ $\left[\dot{\vec{B}}\right]_l$

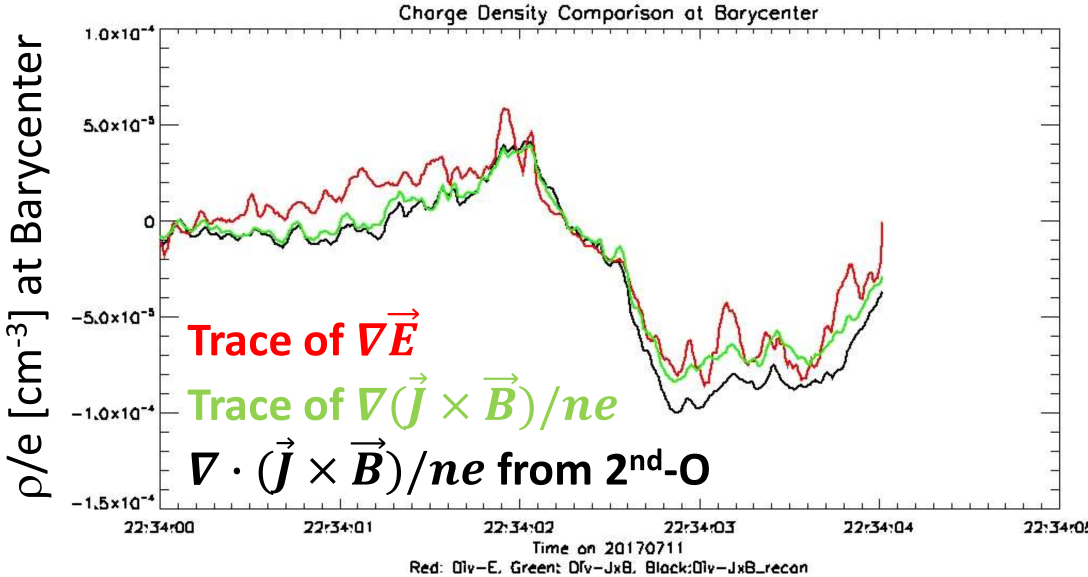
- E reconstruction is possible by applying the same method.
- Subtle differences exist
  $\nabla \cdot \vec{E}$
  $\nabla \times \vec{E}$

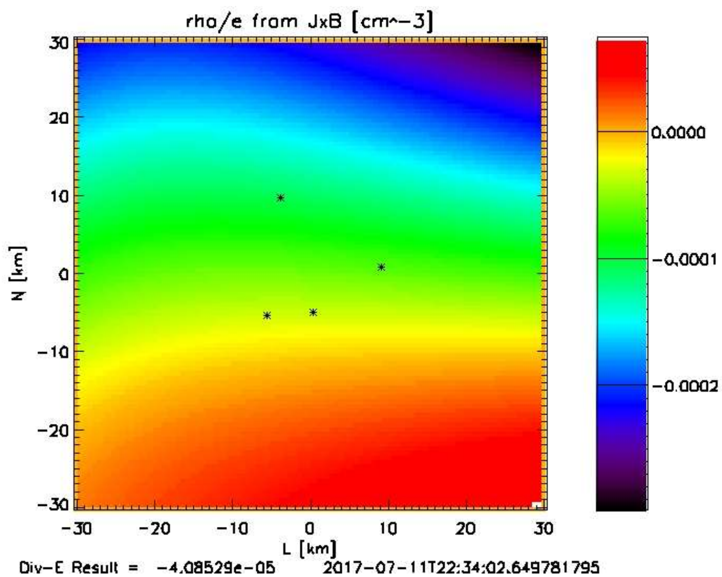
$\vec{E}$ Field Reconstruction Limitations

$$\nabla \cdot \vec{E}$$

- Charge density is a new input parameter.
- It is approximated by the trace of the $\nabla\vec{E}$ tensor.
- Currently, this scalar result is applied throughout the 3D domain.
- This results in increased error in regions where ρ is not constant.

$\nabla \cdot \vec{E}$ in Diffusion Region





$\frac{\rho}{e}$ from Div(E) and Div(JxB/ne) agree well at the barycenter in the diffusion region.

2nd–O Div(JxB/ne) results indicate a gradient in the s/c region.

This leads to E reconstruction errors if large and unaccounted.

Requires a re-work of the constraints: $\nabla\left(\nabla \cdot \vec{E}\right) \neq 0$

Work in progress!

$\vec{E}$ Field Reconstruction Limitations

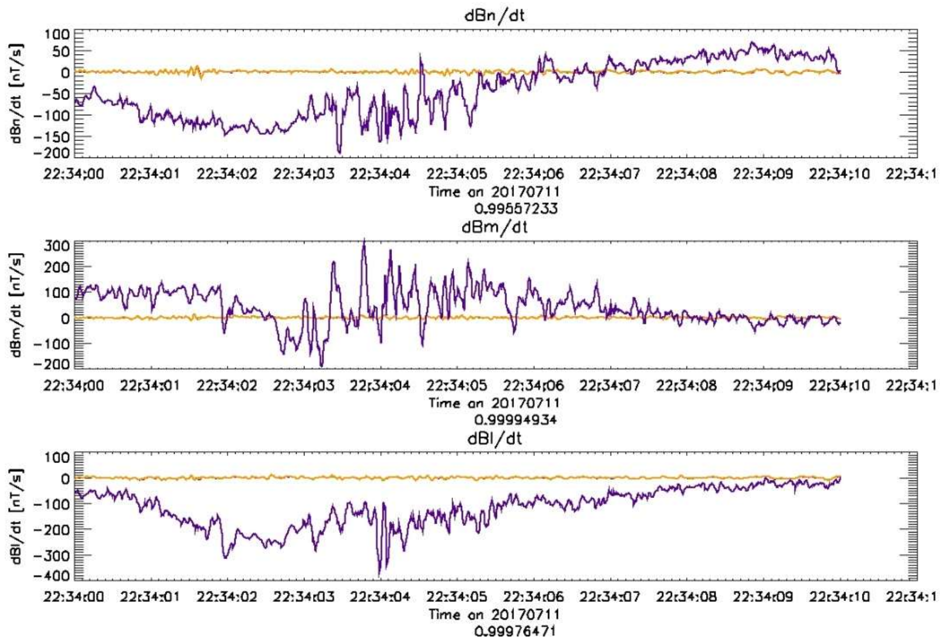
$\nabla \times \vec{E}$

- $\nabla \times \vec{E}$ comes from the off-diagonals of the $\nabla\vec{E}$ tensor.
- $\vec{E}$ measurement uncertainty of the curl is often greater than the true value (poor SNR).
- $\vec{E}$ measurements can be corrected using $\dot{\vec{B}}$.

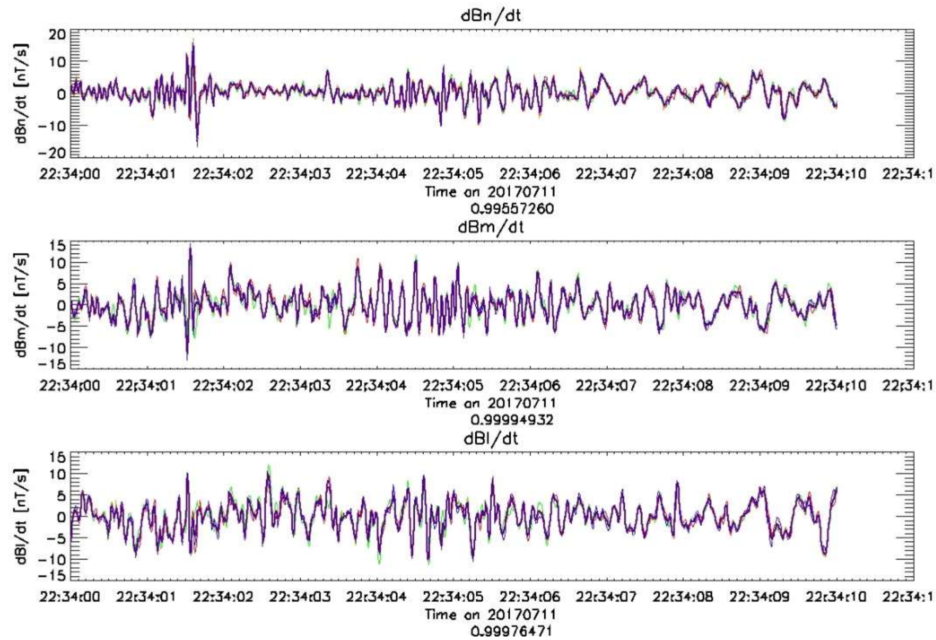
$$\nabla\vec{E} = \begin{bmatrix} \frac{dE_x}{dx} & \frac{dE_y}{dx} & \frac{dE_z}{dx} \\ \frac{dE_x}{dy} & \frac{dE_y}{dy} & \frac{dE_z}{dy} \\ \frac{dE_x}{dz} & \frac{dE_y}{dz} & \frac{dE_z}{dz} \end{bmatrix} = \begin{bmatrix} 0.23 & -0.01 & -1.20 \\ -0.81 & -0.52 & -0.67 \\ -1.95 & -3.26 & -7.09 \end{bmatrix} \rightarrow \begin{bmatrix} 0.23 & -0.41 & -1.57 \\ -0.41 & -0.52 & -1.95 \\ -1.58 & -1.97 & -7.09 \end{bmatrix} [\frac{mV}{10km}]$$

$\nabla \times \vec{E}$ Correction

Uncorrected



Corrected



Purple: $-\nabla \times \vec{E}$ from 1st order

Orange: $\frac{dB}{dt}$ at barycenter

Other colors are $\frac{dB}{dt}$ at s/c

This correction allows for improved reconstruction, especially outside tetrahedron.

Reconstruction Applications

- Have demonstrated the 3D, physics-friendly reconstruction of $\vec{B}$, $\vec{J}$, $\vec{E}$ and $\dot{\vec{B}}$ fields.
- Now other quantities can be evaluated:

$$u = \frac{1}{\mu_0}\vec{B}\cdot\vec{B}$$

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$$

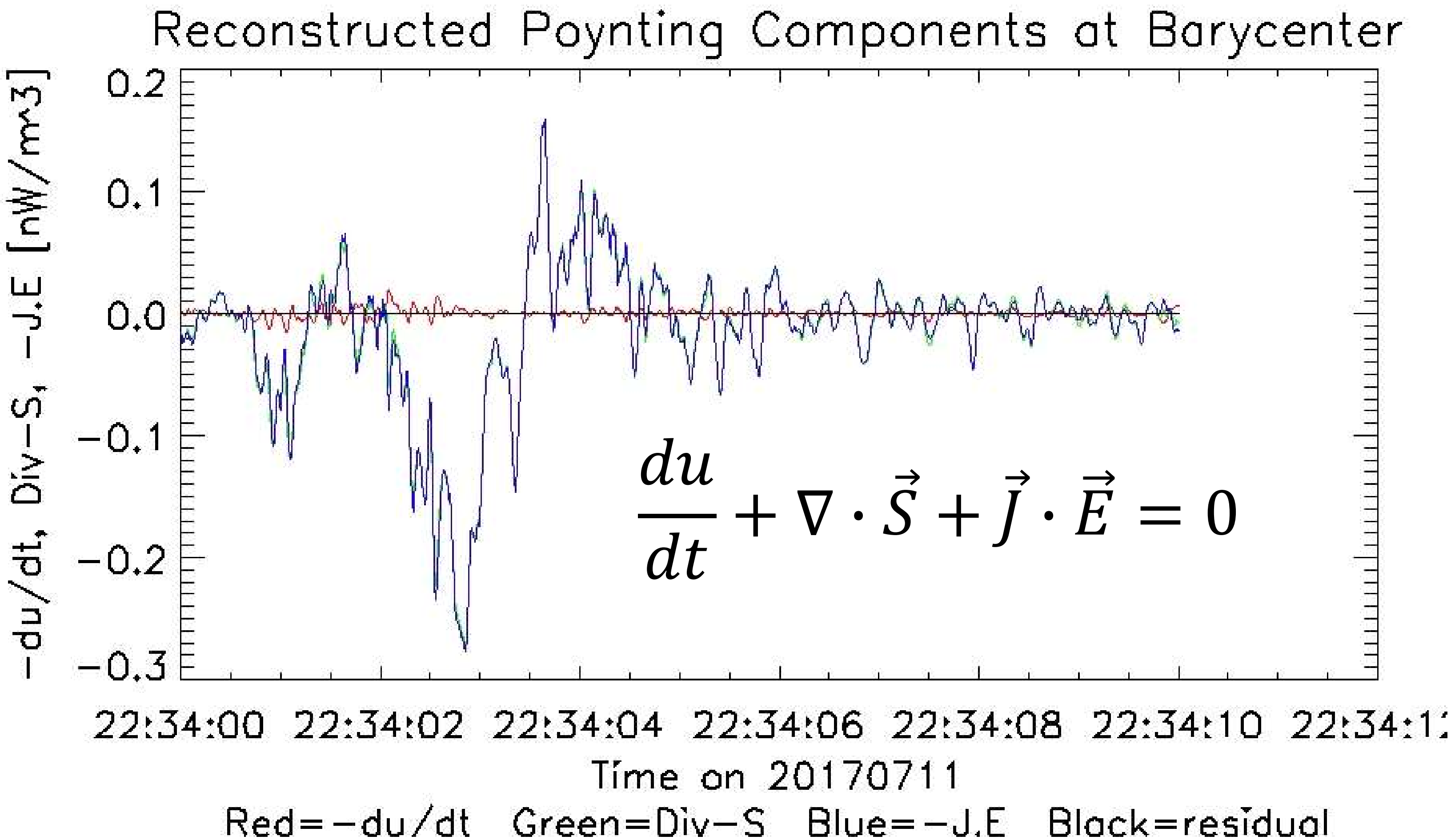
$$\vec{J}\cdot\vec{E}$$

Leads to Poynting's Theorem

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

Particle Tracing

Poynting's Theorem at Barycenter

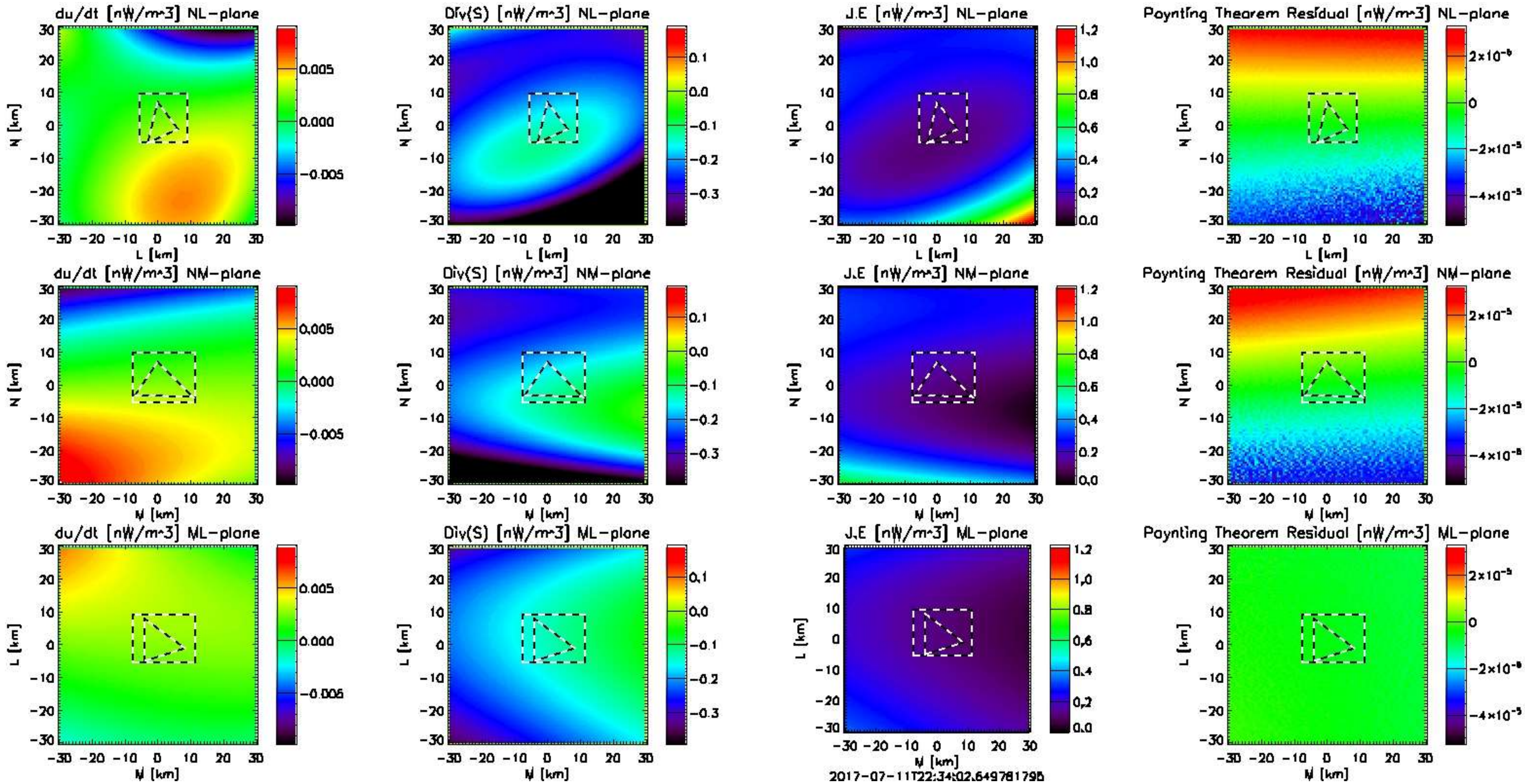


$$\frac{du}{dt} + \nabla \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0$$

Work in progress!

Poynting's Theorem in 3D

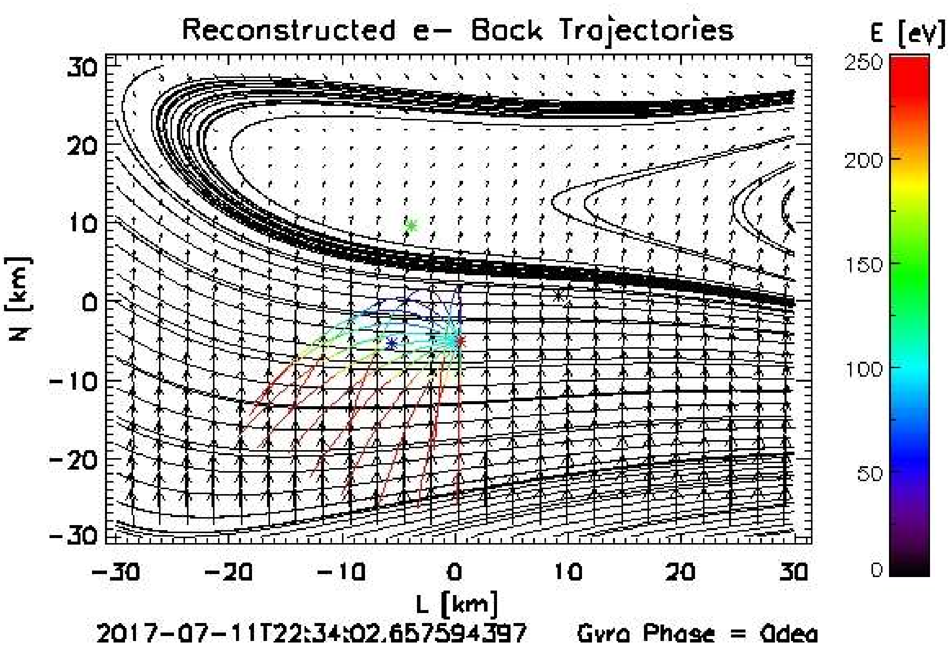
$$\frac{du}{dt} \quad + \quad \nabla \cdot \vec{S} \quad + \quad \vec{J} \cdot \vec{E} \quad = \quad 0$$



Work in progress!

Force Field and Electron Tracing

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$



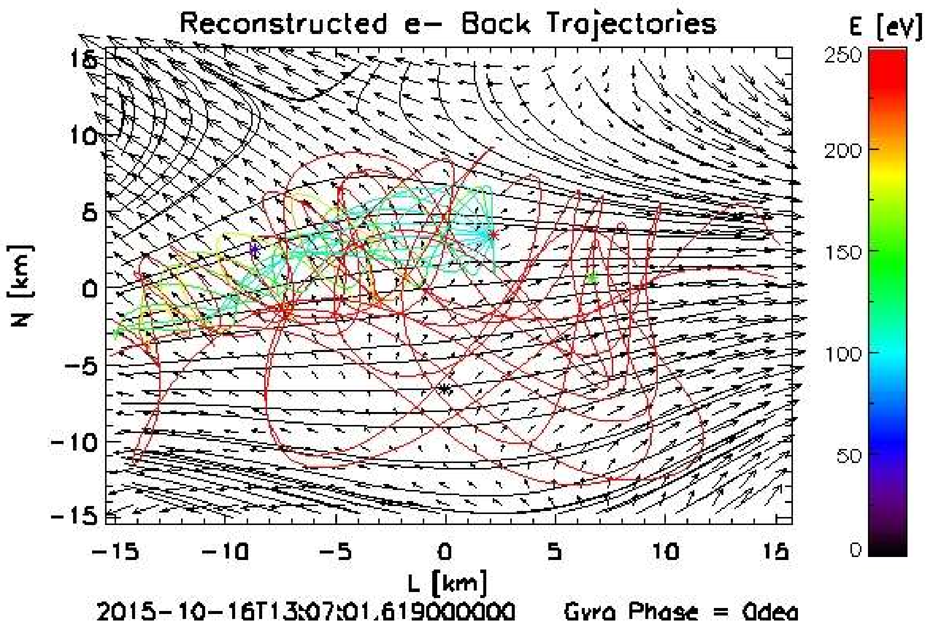
Streamlines: In-plane B
Vector Field: In-plane E

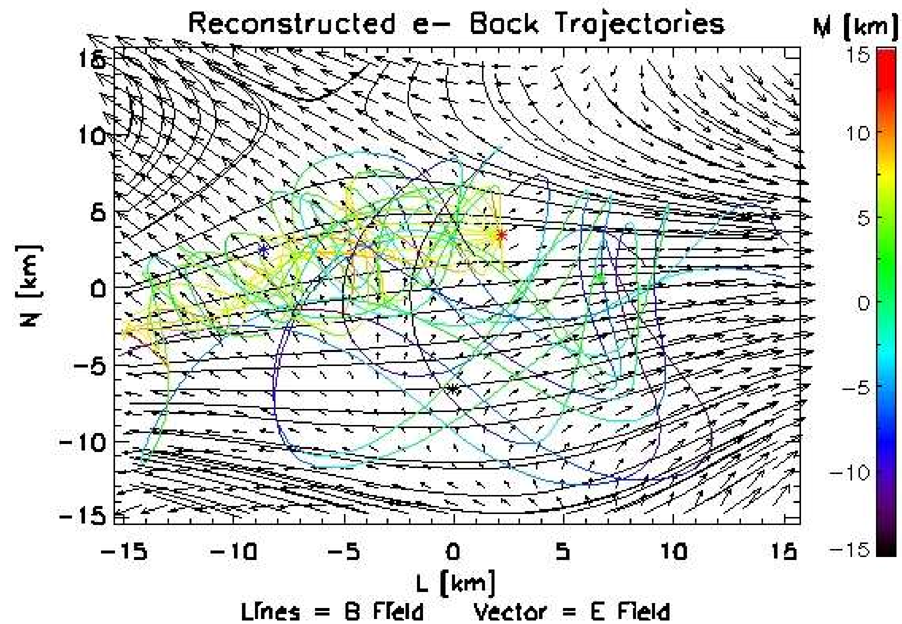
- Force on particles (q,$\vec{v}$) can be calculated.
- Particles measured by s/c can be traced backward to determine source region & energy.
- Plot on left shows 100 eV electrons with a range of pitch angles from MMS2.

Work in progress!

Force Field and Electron Tracing

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$





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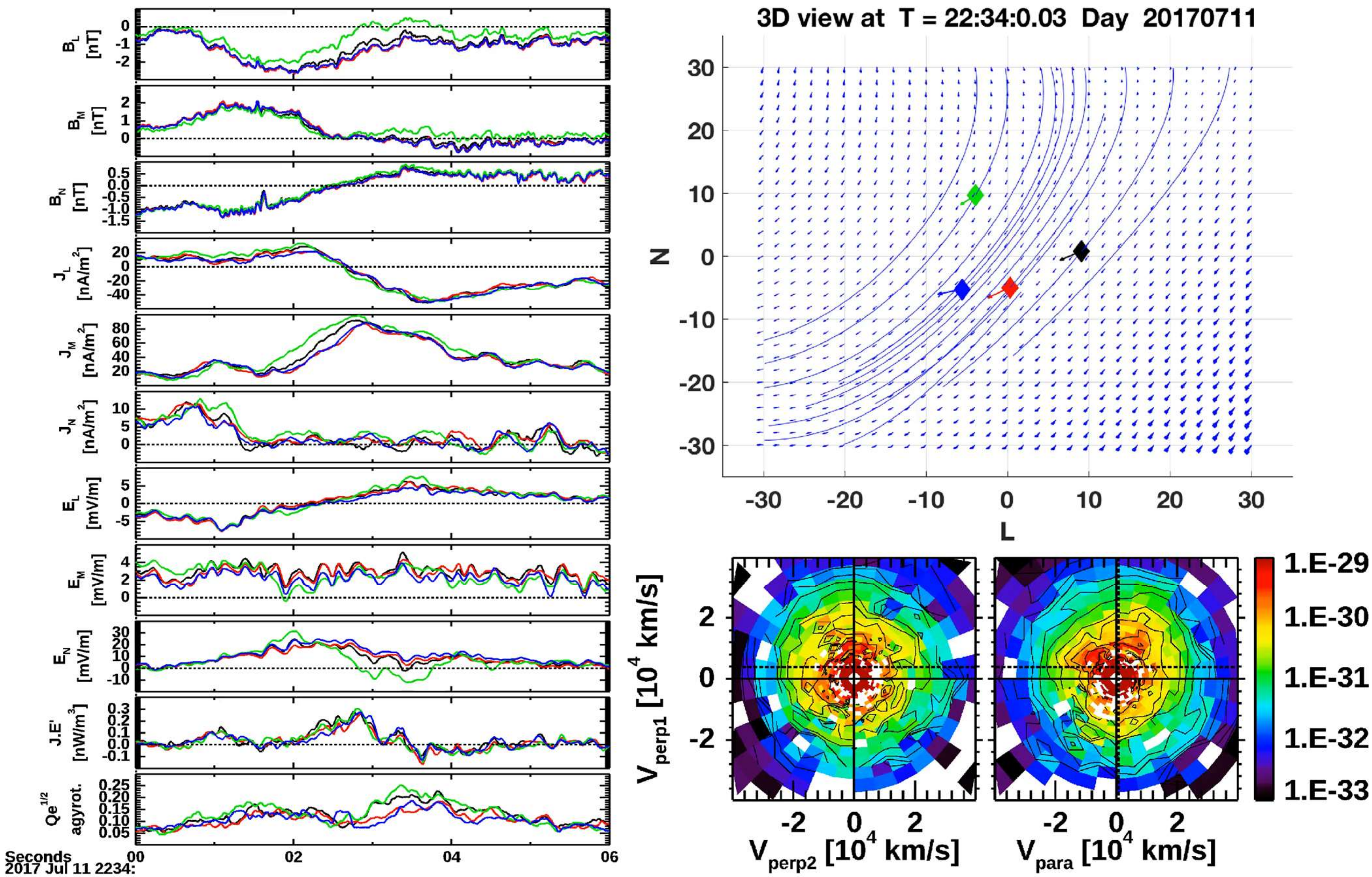
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Summary

- The 2nd order reconstruction method has advantages over 1st order
  - Consistent accuracy in & around tetrahedron
  - Adherence to Maxwell's equations
- These characteristics allow for applications to many 3D problems
  - Poynting's theorem
  - Particle tracing
  - More to come…